

Induced mappings on hyperspaces

著者	Hosokawa Hiroshi
journal or publication title	Tsukuba journal of mathematics
volume	21
number	1
page range	239-250
year	1997-06
URL	http://hdl.handle.net/2241/100431

INDUCED MAPPINGS ON HYPERSPACES

By

Hiroshi HOSOKAWA

Abstract. Let $f : X \rightarrow Y$ be a mapping between continua. Then f induces two mappings $C(f) : C(X) \rightarrow C(Y)$ and $2^f : 2^X \rightarrow 2^Y$ in the natural way. In this paper, we shall study about the following question: Dose the correspondences $f \rightarrow C(f)$ and $f \rightarrow 2^f$ preserve or reverse what classes of mappings? When Y is locally connected, many classes of mappings are preserved by these correspondences. We shall consider the classes of monotone, open, OM, confluent, quasi-monotone and weakly monotone mappings.

1. Introduction

In this paper, continua are compact connected metric spaces, mappings are continuous functions. Throughout this paper, the letters X and Y will always denote nondegenerate continua and a mapping $f : X \rightarrow Y$ is always onto. We shall use the letter d for the metric function for both spaces X and Y . The *hyperspaces* of X are the metric spaces $2^X = \{K \subset X : K \text{ is nonempty and compact}\}$ and $C(X) = \{K \in 2^X : K \text{ is connected}\}$ with the Hausdorff metric H_d (see [8] for the definition of the Hausdorff metric and basic properties of hyperspaces). A mapping $f : X \rightarrow Y$ induces mappings $C(f) : C(X) \rightarrow C(Y)$ and $2^f : 2^X \rightarrow 2^Y$ naturally. If $g : Y \rightarrow Z$ is an another mapping, then $C(g \circ f) = C(g) \circ C(f)$ and $2^{g \circ f} = 2^g \circ 2^f$ hold. Clearly 2^f is onto (since we always assume that $f : X \rightarrow Y$ is onto) but $C(f)$ is onto if and only if f is weakly confluent.

The following three statements for a mapping $f : X \rightarrow Y$ are equivalent:

- (1) f is a homeomorphism;
- (2) $C(f)$ is a homeomorphism;
- (3) 2^f is a homeomorphism.

Key words and phrases: continuum, hyperspace, segment, Whitney map, monotone mapping, open mapping, OM-mapping, confluent mapping, quasi-monotone mapping, weakly monotone mapping, locally connected continuum.

AMS subject classifications (1980): 54B20, 54C05.

Received April 26, 1995

We shall study in the sections below the relations about the above type between the mappings f , $C(f)$ and 2^f .

Some of the results are improvement of those partially appeared in [2] and [3]. But for completeness, we shall describe their proofs.

2. Definitions and Notations

We shall give the list of definitions for mappings treated hereafter. A mapping $f : X \rightarrow Y$ is said to be

- (1) *monotone* if for each $y \in Y$, $f^{-1}(y)$ is connected; equivalently, if for each subcontinuum L of Y , $f^{-1}(L)$ is connected;
- (2) *open* if f maps every open set in X onto an open set in Y ;
- (3) *an OM-mapping* (resp. *an MO-mapping*) if there are mappings g and h , where g is open and h is monotone, such that $f = g \circ h$ (resp. $f = h \circ g$);
- (4) *confluent* if for each subcontinuum L of Y , each component of $f^{-1}(L)$ is mapped by f onto L ;
- (5) *quasi-monotone* if for each subcontinuum L of Y with a nonempty interior, the set $f^{-1}(L)$ has a finite number of components and f maps each of them onto L ;
- (6) *weakly monotone* if for each subcontinuum L of Y with a nonempty interior, each component of the set $f^{-1}(L)$ is mapped by f onto L .

For the implications between these classes of mappings, see p28 in [7].

Let \mathcal{H} denote either $C(X)$ or 2^X . A *Whitney map* $\mu : \mathcal{H} \rightarrow [0, 1]$ is a mapping such that $\mu(\{x\}) = 0$ for each $x \in X$, $\mu(X) = 1$ and if $A, B \in \mathcal{H}$ with $A \subset B \neq A$, then $\mu(A) < \mu(B)$. Such a mapping always exists ([9] or [8]). Let $A_0, A_1 \in \mathcal{H}$. A mapping $\sigma : [0, 1] \rightarrow \mathcal{H}$ is said to be a *segment with respect to the Whitney map μ from A_0 to A_1* provided that $\sigma(0) = A_0$, $\sigma(1) = A_1$, $\mu[\sigma(t)] = (1 - t)\mu(A_0) + t\mu(A_1)$ for each $t \in [0, 1]$ and if $0 \leq t_1 \leq t_2 \leq 1$, then $\sigma(t_1) \subset \sigma(t_2)$. When we use a segment, we will consider it with respect to some fixed Whitney map. A condition of the existence of a segment is as follows:

LEMMA 2.1 ([4] or [8]). *Let $A_0, A_1 \in \mathcal{H}$, where \mathcal{H} denotes either $C(X)$ or 2^X . Then there exists a segment from A_0 to A_1 if and only if*

(2.1.1) $A_0 \subset A_1$ if $\mathcal{H} = C(X)$,

(2.1.2) $A_0 \subset A_1$ and each component of A_1 intersects A_0 if $\mathcal{H} = 2^X$.

Let A_1, A_2, \dots be a sequence of nonempty subsets of X . Then $\liminf A_n$ and $\limsup A_n$ are defined by $\liminf A_n = \{x \in X : \text{if } U \text{ is a neighborhood of } x \text{ in } X,$

then $U \cap A_n \neq \emptyset$ for almost all n , $\limsup A_n = \{x \in X : \text{if } U \text{ is a neighborhood of } x \text{ in } X, \text{ then } U \cap A_n \neq \emptyset \text{ for infinitely many } n\}$. If $\liminf A_n = \limsup A_n = A$, then we say that $\{A_n\}_{n=1}^\infty$ converges to A and write it by $\lim A_n = A$. Following is known:

LEMMA 2.2 [8]. *Let A_1, A_2, \dots be a sequence in 2^X (resp. $C(X)$). Then $\lim A_n = A$ in the sense above if and only if it converges to A with respect to the Hausdorff metric for 2^X (resp. $C(X)$).*

When we say a sequence $\{A_n\}_{n=1}^\infty$ converges in 2^X or $C(X)$, we will mean in a convenient sense of one of the two senses. We shall write \bar{A} , $\text{int } A$ for the closure of A , the interior of A respectively. If \mathcal{A} is a subset of a hyperspace \mathcal{H} , then we shall write $\text{Int } \mathcal{A}$ for the interior of \mathcal{A} in \mathcal{H} .

For a subset A of a space, we say that $A = A_1 \cup A_2$ is a *separation* of A if $A_1 \neq \emptyset \neq A_2$ and $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \emptyset$.

LEMMA 2.3 [10]. *If A and B are nonempty disjoint closed subsets of a compact set K such that no component of K intersects both A and B , then there exists a separation $K = K_a \cup K_b$ of K such that $A \subset K_a$ and $B \subset K_b$.*

Further we shall use the following notation. For any collection \mathcal{A} of subsets of a space, \mathcal{A}^* denotes the union of all members contained in \mathcal{A} .

3. Monotone Mappings

If \mathcal{K} is a subcontinuum of 2^X and $\mathcal{K} \cap C(X) \neq \emptyset$, then \mathcal{K}^* is connected [8]. This is generalized as follows:

LEMMA 3.1. *Let \mathcal{K} be a subcontinuum of 2^X and $K \in \mathcal{K}$. Then each component of \mathcal{K}^* intersects K .*

PROOF. On the contrary, suppose there is a component C of \mathcal{K}^* such that $C \cap K = \emptyset$. Then by lemma 2.3, there is a separation $\mathcal{K}^* = A \cup B$ of \mathcal{K}^* such that $K \subset A$ and $C \subset B$. Put $\mathcal{K}_0 = \{L \in \mathcal{K} : L \subset A\}$ and $\mathcal{K}_1 = \{L \in \mathcal{K} : L \cap B \neq \emptyset\}$. Then we have a separation $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$ of \mathcal{K} . This contradicts to the connectedness of \mathcal{K} .

THEOREM 3.2. *Let $f : X \rightarrow Y$ be a mapping. Then, the following three statements are equivalent:*

- (3.2.1) f is a monotone mapping;
 (3.2.2) $C(f)$ is a monotone mapping;
 (3.2.3) 2^f is a monotone mapping.

PROOF. (3.2.1) \Rightarrow (3.2.2): Suppose that f is monotone and let L be an arbitrary element of $C(Y)$. Put $M = f^{-1}(L)$ and let K be an arbitrary element of $[C(f)]^{-1}(L)$. Then, since f is monotone, M is a subcontinuum of X and contains K . Therefore, by lemma 2.1, there is a segment σ from K to M in $C(X)$. It is evident that the image of σ is contained in $[C(f)]^{-1}(L)$. Thus, in particular, $[C(f)]^{-1}(L)$ is arcwise connected.

(3.2.2) \Rightarrow (3.2.3): Suppose that $C(f)$ is monotone and let B be an arbitrary element of 2^Y . Put $A = f^{-1}(B)$. Then $A \in [2^f]^{-1}(B)$. Let K be a component of A considered as a subset of X . Since $C(f)$ is monotone, $[C(f)]^{-1}(f(K))^*$ is connected and contained in $f^{-1}(f(K))$ and hence is equal to K . Therefore every component of A intersects each element of $[2^f]^{-1}(B)$. It follows by lemma 2.1 that $[2^f]^{-1}(B)$ is arcwise connected.

(3.2.3) \Rightarrow (3.2.1): Suppose that 2^f is monotone and let $y \in Y$. Then by lemma 3.1, $[2^f]^{-1}(\{y\})^* = f^{-1}(y)$ is connected.

REMARK. If f is monotone and \mathcal{B} is an arcwise connected subcontinuum of 2^Y (resp. $C(Y)$), then $[2^f]^{-1}(\mathcal{B})$ (resp. $[C(f)]^{-1}(\mathcal{B})$) is arcwise connected.

4. Open Mappings

The following lemma is a characterization of open mappings. The equivalence (4.1.1.) \Leftrightarrow (4.1.2) is appeared in [7], p.14 without proof (see also [5], pp. 67–68).

LEMMA 4.1. Let $f : X \rightarrow Y$ be a mapping. Then the following three statements are equivalent:

- (4.1.1) f is an open mapping;
 (4.1.2) for each sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $\lim y_n = y$, $\limsup f^{-1}(y_n) = f^{-1}(y)$;
 (4.1.3) for each sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $\lim y_n = y$, $\{f^{-1}(y_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(y)$.

PROOF. The implication (4.1.3) \Rightarrow (4.1.2) is evident.

(4.1.1) \Rightarrow (4.1.3): Suppose f is open and let $\{y_n\}_{n=1}^{\infty}$ be a sequence in Y such that $\lim y_n = y$. Since the continuity of f implies $\limsup f^{-1}(y_n) \subset f^{-1}(y)$, it is

sufficient to show that $f^{-1}(y) \subset \liminf f^{-1}(y_n)$. Let $x \in f^{-1}(y)$ and U an open neighborhood of x in X . Since $f(U)$ is a neighborhood of y , there is an integer n_0 such that $y_n \in f(U)$ and hence $f^{-1}(y_n) \cap U \neq \emptyset$ for each $n \geq n_0$. Therefore $x \in \liminf f^{-1}(y_n)$ and hence we have $f^{-1}(y) \subset \liminf f^{-1}(y_n)$.

For any collection U_1, U_2, \dots, U_n of open sets in X , let $\langle U_1, U_2, \dots, U_n \rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^n U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, 2, \dots, n\}$. It is known that:

LEMMA 4.2 [8]. *The collection of all subsets of 2^X of the form $\langle U_1, U_2, \dots, U_n \rangle$ is a base for the Hausdorff metric topology for 2^X .*

THEOREM 4.3. *Let $f : X \rightarrow Y$ be a mapping. Consider the following three statements:*

(4.3.1) *f is an open mapping;*

(4.3.2) *$C(f)$ is an open mapping;*

(4.3.3) *2^f is an open mapping.*

Then (4.3.1) and (4.3.3) are equivalent and (4.3.2) implies (4.3.1).

PROOF. (4.3.1) \Rightarrow (4.3.3): Suppose f is open and let $\{B_n\}_{n=1}^\infty$ be a sequence in 2^Y such that $\lim B_n = B$. Since 2^f is continuous, $\limsup [2^f]^{-1}(B_n)$ is contained in $[2^f]^{-1}(B)$. Let A be an arbitrary element of $[2^f]^{-1}(B)$ and let U_1, U_2, \dots, U_r be open sets in X such that $A \in \langle U_1, U_2, \dots, U_r \rangle$. Since A is compact, there are open sets V_1, V_2, \dots, V_r of X such that $\bar{V}_i \subset U_i$ for each $i = 1, 2, \dots, r$ and $A \in \langle V_1, V_2, \dots, V_r \rangle$. Since f is open, $\langle f(V_1), f(V_2), \dots, f(V_r) \rangle$ is an open neighborhood of $f(A) = B$ in 2^Y . Therefore there is an integer n_0 such that $B_n \in \langle f(V_1), f(V_2), \dots, f(V_r) \rangle$ for each $n \geq n_0$. Put $A_n = f^{-1}(B_n) \cap [\bigcup_{i=1}^r \bar{V}_i]$. Then it is easy to see that $A_n \in [2^f]^{-1}(B_n) \cap \langle U_1, U_2, \dots, U_r \rangle$ and hence by lemma 4.2, we have $A \in \liminf [2^f]^{-1}(B_n)$. It follows from lemma 4.1, that 2^f is an open mapping.

(4.3.3) \Rightarrow (4.3.1): Suppose 2^f is an open mapping. Let U be an open set in X and let $x \in U$. Since $\langle U \rangle$ is an open neighborhood of $\{x\} \in 2^X$, $2^f(\langle U \rangle) = \langle f(U) \rangle$ is an open neighborhood of $\{f(x)\} \in 2^Y$. Therefore $f(U)$ is a neighborhood of $f(x)$. Since x is an arbitrary element of U , $f(U)$ is open in Y .

The proof of the implication (4.3.2) \Rightarrow (4.3.1) is similar.

Note that in general, $C(f)([\langle U_1, U_2, \dots, U_n \rangle] \cap C(X))$ is not equal to $\langle f(U_1), f(U_2), \dots, f(U_n) \rangle \cap C(Y)$ even though $n = 1$. Following is an example where f is open, X and Y are locally connected but $C(f)$ is not open.

EXAMPLE Let X, Y be plane continua defined by

$$Y = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\},$$

$$X = \{(x, y) : (x, y) \in Y \text{ or } (-x, -y) \in Y\}.$$

Define $f : X \rightarrow Y$ by

$$f(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in Y \\ (-x, -y) & \text{if } (x, y) \notin Y. \end{cases}$$

for each $(x, y) \in X$. Let $K = \{(x, y) \in X : x = 0 \text{ or } y = 0\}$. Then f is open but $C(f)$ is not open at $K \in C(X)$.

5. OM-mappings

In [6], A. Lelek and D. R. Read had given a characterization of OM-mappings as follows:

LEMMA 5.1 [6]. *A mapping $f : X \rightarrow Y$ is an OM-mapping if and only if for each $y \in Y$ and each sequence $\{y_n\}_{n=1}^{\infty}$ in Y , $\lim y_n = y$ implies that $\limsup f^{-1}(y_n)$ meets each component of $f^{-1}(y)$.*

We always saw that the correspondence $f \rightarrow C(f)$ does not preserve the class of open mappings. Nevertheless it preserves the class of OM-mappings.

THEOREM 5.2. *For a mapping $f : X \rightarrow Y$, the following three statements are equivalent:*

- (5.2.1) f is an OM-mapping;
- (5.2.2) $C(f)$ is an OM-mapping;
- (5.2.3) 2^f is an OM-mapping.

PROOF. The implication (5.2.1) \Rightarrow (5.2.3) follows from Theorems 3.2 and 4.3.

(5.2.1) \Rightarrow (5.2.2): Suppose f is an OM-mapping and $\{L_n\}_{n=1}^{\infty}$ is a sequence in $C(Y)$ which converges to $L \in C(Y)$. Let \mathcal{K} be a component of $[C(f)]^{-1}(L)$. We must show that $\limsup [C(f)]^{-1}(L_n) \cap \mathcal{K} \neq \emptyset$. Choose a point $x \in \mathcal{K}^*$ and put $y = f(x)$. There is a point $y_n \in L_n$ for each $n = 1, 2, \dots$ such that $\lim y_n = y$. Let C be the component of $f^{-1}(y)$ containing x . Since f is an OM-mapping, there is a point $x_n \in f^{-1}(y_n)$ such that some subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to some point of C . We may assume $\lim x_n = x_0 \in C$. Let K_n be the component of

$f^{-1}(L_n)$ containing x_n for each $n = 1, 2, \dots$. Since OM-mappings are confluent, we have $K_n \in [C(f)]^{-1}(L_n)$. We may assume that $\{K_n\}_{n=1}^\infty$ converges to K_0 for some $K_0 \in C(X)$. It is easy to see that K_0 and $K_0 \cup C$ are elements of $C(X)$ contained in the same component of $[C(f)]^{-1}(L)$. Let K be an element of \mathcal{K} such that $x \in K$. Then K and $K_0 \cup C$ are in the same component of $[C(f)]^{-1}(L)$. Thus $K_0 \in \mathcal{K}$ and hence we have $\limsup [C(f)]^{-1}(L_n) \cap \mathcal{K} \neq \emptyset$. Therefore by lemma 5.1, $C(f)$ is an OM-mapping.

(5.2.2) \Rightarrow (5.2.1): Suppose $C(f)$ is an OM-mapping and $\{y_n\}_{n=1}^\infty$ is a sequence in Y which converges to $y \in Y$. Clearly the sequence $\{\{y_n\}_{n=1}^\infty\}$ considered as a sequence in $C(Y)$, converges to $\{y\} \in C(Y)$. Let K be a component of $f^{-1}(y)$. Then $C(K)$, considered as a subset of $C(X)$, is a component of $[C(f)]^{-1}(\{y\})$. By the assumption and lemma 5.1, there is $K_n \in [C(f)]^{-1}(\{y_n\})$ for each n such that some subsequence of $\{K_n\}_{n=1}^\infty$ converges to an element of $C(K)$. Since $K_n \subset f^{-1}(y_n)$, this implies that $\limsup f^{-1}(y_n) \cap K \neq \emptyset$. Therefore applying lemma 5.1 again, we have that f is an OM-mapping.

The implication (5.2.3) \Rightarrow (5.2.1) is similarly proved.

THEOREM 5.3. *If $f : X \rightarrow Y$ is an MO-mapping, then 2^f is also an MO-mapping.*

This follows directly from Theorems 3.2 and 4.3.

6. Confluent mappings

First we prove a special case.

LEMMA 6.1. *Let $f : X \rightarrow Y$ be a confluent mapping.*

(6.1.1) *If \mathcal{L} is an arc in $C(Y)$, then each component of $[C(f)]^{-1}(\mathcal{L})$ is mapped by $C(f)$ onto \mathcal{L} .*

(6.1.2) *If \mathcal{L} is an arc in 2^Y , then each component of $[2^f]^{-1}(\mathcal{L})$ is mapped by 2^f onto \mathcal{L} .*

PROOF. We only prove (6.1.2) since (6.1.1) is more simple. Let \mathcal{L} be an arc in 2^Y and $\alpha : [0, 1] \rightarrow \mathcal{L}$ a homeomorphism. Let \mathcal{K} be a component of $[2^f]^{-1}(\mathcal{L})$. Without loss of generality, we may assume $\alpha(0) \in 2^f(\mathcal{K})$. It is sufficient to show that $\alpha(1) \in 2^f(\mathcal{K})$. On the contrary, suppose that $\alpha(1) \notin 2^f(\mathcal{K})$. Then by lemma 2.3, there is a separation $[2^f]^{-1}(\mathcal{L}) = \mathcal{K}_0 \cup \mathcal{K}_1$

such that $\mathcal{K} \subset \mathcal{K}_0$ and $[2^f]^{-1}(\alpha(1)) \subset \mathcal{K}_1$. Put $t_0 = \sup\{t : \alpha(t) \in 2^f(\mathcal{K}_0)\}$. Then by compactness of \mathcal{K}_0 , $t_0 < 1$ and there is $K \in \mathcal{K}_0$ such that $2^f(K) = \alpha(t_0)$. Let M be the union of all components C of $f^{-1}(\alpha(t_0))$ such that $C \cap K \neq \emptyset$. Note that $M \in \mathcal{K}_0$ since K and M are joined by a segment in $[2^f]^{-1}(\alpha(t_0))$. Let M_t be the union of all components of $f^{-1}(\alpha([t_0, t])^*)$ intersecting M for each $t \in [t_0, 1]$. Choose a sequence t_1, t_2, \dots in $[t_0, 1]$ such that $1 > t_1 > t_2 > \dots$, and $\lim t_n = t_0$. For each $n = 1, 2, \dots$, put $K_n = f^{-1}(\alpha(t_n)) \cap M_{t_n}$. Since f is confluent, each component of M_{t_n} is mapped by f onto a component of $\alpha([t_0, t_n])^*$. Therefore, by lemma 3.1, it is not so difficult to see that $K_n \in [2^f]^{-1}(\alpha(t_n))$ and each component of M_{t_n} intersects K_n . We may assume that $\lim K_n = K_0$ for some $K_0 \in 2^X$. Then $K_0 \subset \bigcap_{n=1}^{\infty} M_{t_n} = M$ and each component of M intersects K_0 . Therefore by lemma 2.1, there is a segment from K_0 to M whose image is clearly contained in $[2^f]^{-1}(\alpha(t_0))$. Therefore $K_0 \in \mathcal{K}_0$. On the other hand, $K_n \in \mathcal{K}_1$ for each $n = 1, 2, \dots$. Hence we have a contradiction since $H_d(\mathcal{K}_0, \mathcal{K}_1) > 0$.

COROLLARY 6.2. *Let $f : X \rightarrow Y$ be a confluent mapping.*

(6.2.1) *If \mathcal{L} is an arcwise connected subcontinuum of $C(Y)$, then each component of $[C(f)]^{-1}(\mathcal{L})$ is mapped by $C(f)$ onto \mathcal{L} .*

(6.2.2) *If \mathcal{L} is an arcwise connected subcontinuum of 2^Y , then each component of $[2^f]^{-1}(\mathcal{L})$ is mapped by 2^f onto \mathcal{L} .*

PROOF. Let \mathcal{L} be an arcwise connected subcontinuum of $C(Y)$ and let \mathcal{K} be a component of $[C(f)]^{-1}(\mathcal{L})$. Choose an element $K \in \mathcal{K}$. Then for any $L \in \mathcal{L} - \{f(K)\}$, there is an arc \mathcal{B} in \mathcal{L} with the end points $f(K)$ and L . Let \mathcal{A} be the component of $[C(f)]^{-1}(\mathcal{B})$ containing K . Then clearly $\mathcal{A} \subset \mathcal{K}$, lemma 6.1 implies $L \in C(f)(\mathcal{K})$. (6.2.2) is similarly proved.

THEOREM 6.3. *Let $f : X \rightarrow Y$ be a mapping. Consider the following three statements:*

(6.3.1) *f is a confluent mapping;*

(6.3.2) *$C(f)$ is a confluent mapping;*

(6.3.3) *2^f is a confluent mapping.*

Then the implications (6.3.2) \Rightarrow (6.3.1) and (6.3.3) \Rightarrow (6.3.1) hold. If Y is locally connected, then they are equivalent.

PROOF. (6.3.3) \Rightarrow (6.3.1): Let L be a subcontinuum of Y and K a component of $f^{-1}(L)$. Let \mathcal{L} and \mathcal{K} be subcontinua of 2^Y and 2^X respectively

defined by $\mathcal{L} = \{\{y\} : y \in L\}$, $\mathcal{K} = \{\{x\} : x \in K\}$. Let \mathcal{M} be a component of $[2^f]^{-1}(\mathcal{L})$ such that $\mathcal{K} \cap \mathcal{M} \neq \emptyset$. Then it is clear that $\mathcal{M}^* = K$. Since $2^f(\mathcal{M}) = \mathcal{L}$, we have $f(K) = L$.

The implication (6.3.2) \Rightarrow (6.3.1) is similarly proved.

Now suppose that f is confluent and Y is locally connected. We shall only prove that 2^f is confluent and omit the proof for $C(f)$ to be confluent. Let \mathcal{L} be a subcontinuum of 2^X and \mathcal{K} a component of $[2^f]^{-1}(\mathcal{L})$. Since 2^Y is locally connected ([1] or [8]), there are locally connected subcontinua $\mathcal{L}_1, \mathcal{L}_2, \dots$ of 2^Y such that $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} \mathcal{L}_n = \mathcal{L}$ (see [5], p.260). Let \mathcal{K}_n be the component of $[2^f]^{-1}(\mathcal{L}_n)$ containing \mathcal{K} for each $n = 1, 2, \dots$. It follows evidently that $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} \mathcal{K}_n = \mathcal{K}$. Since by corollary 6.2 and continuity of 2^f , $2^f(\mathcal{K}) = 2^f(\bigcap_{n=1}^{\infty} \mathcal{K}_n) = \bigcap_{n=1}^{\infty} 2^f(\mathcal{K}_n) = \bigcap_{n=1}^{\infty} \mathcal{L}_n = \mathcal{L}$.

The following example shows that there is a confluent mapping f such that neither $C(f)$ nor 2^f is weakly confluent.

EXAMPLE. In the Euclidean plane with polar coordinates (r, θ) , let S be the unite circle $S = \{(r, \theta) : r = 1 \text{ and } 0 \leq \theta < 2\pi\}$ and let A_1, A_2, B_1, B_2 be spaces each homeomorphic to the half open interval $[0, 1)$, defined by

$$\begin{aligned} A_1 &= \left\{ (r, \theta) : \theta = \frac{\pi}{2} \sin \frac{1}{1-r}, 1 < r \leq 2 \right\}, \\ A_2 &= \left\{ (r, \theta) : \theta = \frac{\pi}{2} \left(2 + \sin \frac{1}{1-r} \right), \frac{1}{2} \leq r < 1 \right\}, \\ B_1 &= \left\{ (r, \theta) : \theta = \pi \sin \frac{1}{1-r}, 1 < r \leq 2 \right\}, \\ B_2 &= \left\{ (r, \theta) : \theta = \pi \left(2 + \sin \frac{1}{1-r} \right), \frac{1}{2} \leq r < 1 \right\}. \end{aligned}$$

Define X, Y and $f : X \rightarrow Y$ by $X = S \cup A_1 \cup A_2$, $Y = S \cup B_1 \cup B_2$ and $f(r, \theta) = (r, 2\theta)$ for all $(r, \theta) \in X$. Then f is confluent and weakly monotone. Let $K_t = \{(r, \theta) : r = 1 \text{ and } (\pi/2)(t-1) \leq \theta \leq (\pi/2)(2t-1)\}$ for $t \in [0, 1]$ and $L_t = \{(r, \theta) : r = 1 \text{ and } (\pi/2)(t+1) \leq \theta \leq (\pi/2)(2t+1)\}$ for $t \in [0, 1]$. The sets $\mathcal{K} = \{K_t : t \in [0, 1]\}$ and $\mathcal{L} = \{L_t : t \in [0, 1]\}$ are disjoint arcs in $C(X)$ such that $C(f)(\mathcal{K}) = C(f)(\mathcal{L})$. There exist subsets \mathcal{M}, \mathcal{N} of $C(X)$ such that \mathcal{M} and \mathcal{N} are both homeomorphic to the half open interval, each element of \mathcal{M} (resp. \mathcal{N}) is contained in A_1 (resp. A_2), $\overline{\mathcal{M}} - \mathcal{M} = \mathcal{K}$ and $\overline{\mathcal{N}} - \mathcal{N} = \mathcal{L}$. To see this, let $g : S \cup A_1 \rightarrow S$ be the retraction defined by $g(r, \theta) = (1, \theta)$ for each $(r, \theta) \in S \cup A_1$. We consider $C(X)$ and $C(S \cup A_1)$ as subsets of $C(X)$. Put $\mathcal{M}_0 = [C(g)]^{-1}(\mathcal{K})$.

Note that $[C(g)]^{-1}(K_1) - \{K_1\}$ is a disjoint union of countably many arcs in $C(S \cup A_1)$ and $[C(g)]^{-1}(K_t)$ is a countable set with one limit element K_t for $0 \leq t < 1$. Define $\mathcal{M} = \mathcal{M}_0 - \mathcal{K}$. Similarly we can find a described set \mathcal{N} . Put $\mathcal{A}_1 = \mathcal{M} \cup \mathcal{K}$, $\mathcal{A}_2 = \mathcal{N} \cup \mathcal{L}$ and $\mathcal{B} = C(f)(\mathcal{A}_1 \cup \mathcal{A}_2)$. Then \mathcal{B} is a subcontinuum of $C(Y)$ and $[C(f)]^{-1}(\mathcal{B})$ has just two components \mathcal{A}_1 and \mathcal{A}_2 . But neither of them is mapped by $C(f)$ onto \mathcal{B} .

7. Quasi-monotone and Weakly monotone mappings

LEMMA 7.1. *If $f: X \rightarrow Y$ is weakly monotone and Y is locally connected, then f is confluent.*

PROOF. Let L be a subcontinuum of Y and K a component of $f^{-1}(L)$. Since Y is locally connected, there are subcontinua $L_n (n = 1, 2, \dots)$ of Y such that $L_1 \supset L_2 \supset L_3 \supset \dots$, $\bigcap_{n=1}^{\infty} L_n = L$ and $\text{int } L_n \neq \emptyset$ for each $n = 1, 2, \dots$ (see [5] or [10]). Let K_n be a component of $f^{-1}(L_n)$ containing K for each $n = 1, 2, \dots$. Then clearly $K = \bigcap_{n=1}^{\infty} K_n$ and hence $f(K) = \bigcap_{n=1}^{\infty} f(K_n) = \bigcap_{n=1}^{\infty} L_n = L$.

By Theorem 6.3, we have:

COROLLARY 7.2. *If $f: X \rightarrow Y$ is weakly monotone and Y is locally connected, then both of the mappings $C(f)$ and 2^f are confluent.*

THEOREM 7.3. *Let $f: X \rightarrow Y$ be a mapping. Consider the following three statements:*

- (7.3.1) *f is a quasi-monotone (resp. a weakly monotone) mapping;*
- (7.3.2) *$C(f)$ is a quasi-monotone (resp. a weakly monotone) mapping;*
- (7.3.3) *2^f is a quasi-monotone (resp. a weakly monotone) mapping.*

Then one of (7.3.2) and (7.3.3) implies (7.3.1). If Y is locally connected, then they are equivalent.

PROOF. We shall only prove for the class of quasi-monotone mappings. The proof of the implication that (7.3.2) or (7.3.3) implies (7.3.1) is similar as the proof of Theorem 4.3.

Now suppose f is quasi-monotone and Y is locally connected. Let \mathcal{L} be a subcontinuum of $C(Y)$ such that $\text{Int } \mathcal{L} \neq \emptyset$. Choose $L_0 \in \text{Int } \mathcal{L}$ and $y \in L_0$. Since $L_0 \in \text{Int } \mathcal{L}$ and Y is locally connected, there is a small closed connected neighborhood V of y in Y such that $L = V \cup L_0 \in \mathcal{L}$. Since f is quasi-monotone and $\text{int } L \neq \emptyset$, $f^{-1}(L)$ has a finite number of components, say K_1, K_2, \dots, K_r ,

each of them is mapped by f onto L . Since quasi-monotone mappings are weakly monotone, corollary 7.2 implies that $C(f)$ is confluent. Let \mathcal{K}_0 be a component of $[C(f)]^{-1}(\mathcal{L})$. Then $C(f)(\mathcal{K}_0) = \mathcal{L}$. Thus there is $K \in \mathcal{K}_0$ such that $C(f)(K) = L$. Therefore $K \subset K_i$ for some $i \in \{1, 2, \dots, r\}$. Then by lemma 2.1, it is easy to see that $K_i \in \mathcal{K}_0$. Therefore the number of components of $[C(f)]^{-1}(\mathcal{L})$ is at most r . This and corollary 7.2 implies that $C(f)$ is quasi-monotone.

Next, suppose that $f : X \rightarrow Y$ is quasi-monotone and Y is locally connected (the case for weakly monotone mappings are follows from corollary 7.2). By Theorem 6.3, 2^f is confluent. Let \mathcal{B} be a subcontinuum of 2^Y with a nonempty interior and $B \in \text{Int } \mathcal{B}$. There is a positive number ε such that if $L \in 2^Y$ and $H_d(B, L) < \varepsilon$, then $L \in \mathcal{B}$. Since Y is uniformly locally connected, there are $\delta > 0$ and $M \in 2^Y$ such that $V_\delta(B) \subset M \subset V_\varepsilon(B)$ and each component of M intersects B , where $V_\gamma(B)$ is the γ -neighborhood of B in Y for each $\gamma > 0$ (see [10], pp. 20–22). The number of components of M is finite because let $\{M_\alpha : \alpha \in \Omega\}$ be the set of components of M , choose a point $y_\alpha \in M_\alpha \cap B$ for each $\alpha \in \Omega$, then the set $\{y_\alpha : \alpha \in \omega\}$ is discrete and hence a finite set. Let M_1, M_2, \dots, M_r be the components of M . Since f is quasi-monotone and $\text{int } M_i \neq \emptyset$, $f^{-1}(M_i)$ has finitely many, say $n(i)$, components for each $i = 1, 2, \dots, r$. Then, as the proof of (7.3.1) \Rightarrow (7.3.2), the number of components of $[2^f]^{-1}(\mathcal{B})$ is at most $n(1) \cdot n(2) \dots n(r)$.

8. Problems

There is an open mapping f such that $C(f)$ is not open (the example in section 4).

1. Is there an open mapping $f : X \rightarrow Y$ such that $C(f)$ is open but $C(C(f))$ is not open?

2. Does the correspondence $f \rightarrow C(f)$ preserve or reverse the class of MO-mappings? If f is open, then is $C(f)$ an MO-mapping? If 2^f is an MO-mapping, then is f an MO-mapping?

3. For a confluent mapping $f : X \rightarrow Y$, is it true that if 2^f is confluent, then $C(f)$ is confluent?

A continuum X is said to have property [K] if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $a, b \in X$, $d(a, b) < \delta$ and $a \in A \in C(X)$, then there exists $B \in C(X)$ such that $b \in B$ and $H_d(A, B) < \varepsilon$.

It is easy to see that if \mathcal{A} is a subcontinuum of 2^X and $\text{Int } \mathcal{A} \neq \emptyset$, then $\text{int } \mathcal{A}^* \neq \emptyset$. If X has property [K], then for a subcontinuum \mathcal{K} of $C(X)$,

$\text{Int } \mathcal{K} \neq \phi$ implies $\text{int } \mathcal{K}^* \neq \phi$. But if X does not have property [K], $\text{int } \mathcal{K}^*$ may be empty.

EXAMPLE. In the Euclidean plan, let us denote xy the straight line segment with the end points x, y . Let $p = (1, 0)$, $q = (-1, 0)$ and $a_n = (0, 1/n)$ for each $n = 1, 2, \dots$. Let $A_n = a_{2n}p$, $B_n = a_{2n+1}q$ for $n = 1, 2, \dots$ and $C = pq$. Let $X = C \cup [\bigcup_{n=1}^{\infty} A_n] \cup [\bigcup_{n=1}^{\infty} B_n]$ and $\mathcal{K} = \{p_s p_t : s - t = 1 \text{ and } 1/3 \leq t \leq 2/3\}$, where $p_s = (s, 0) \in X$, then \mathcal{K} is a subcontinuum of $C(X)$ such that $\text{Int } \mathcal{K} \neq \phi$ but $\text{int } \mathcal{K}^* = \phi$.

4. In Theorem 6.3, can the condition “ Y is locally connected” be weakend?

Added in proof H. Kato announced me that by adding countably many disjoint half open lines on the continua of the example in section 6 of this paper, it is possible to construct continua having property [K] and a confluent mapping between them whose induced mappings are not weakly confluent.

Recently A. Illanes answered Problem 1 affirmatively. He showed that if $C(C(f))$ is open, then f is a homeomorphism.

References

- [1] D. W. Curtis and R. M. Schori, 2^X and $C(X)$ are homeomorphic to the Hilbert cube, Bull. Amer. Math. Soc., **80** (1974), pp. 927–931.
- [2] H. Hosokawa, Induced mappings between hyperspaces, Bull. Tokyo Gakugei Univ., Sect. 4, **41** (1989), pp. 1–6.
- [3] H. Hosokawa, Induced mappings between hyperspaces II, Bull. Tokyo Gakugei Univ., Sect. 4, **44** (1992), pp. 1–7.
- [4] J. L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc., **52** (1942), pp. 22–36.
- [5] K. Kuratowski, Topology, vol II, New York-London-Warszawa, 1966.
- [6] A. Lelek and D. R. Read, Compositions of confluent mappings and some other classes of functions, Colloq. Math. **29** (1974), pp. 101–112.
- [7] T. Mackowiak, Continuous mappings on continua, Dissert. Math. (Rozprawy Matem.), 158, Warszawa (1979).
- [8] S. B. Nadler, Jr., Hyperspaces of sets, Marcel Dekker, Inc., New York (1978).
- [9] H. Whitney, Regular families of curves, Annals of Math., **34** (1993), pp. 244–270.
- [10] G. T. Whyburn, Analytic topology, New York. (1942).

Department of Mathematics,
Tokyo Gakugei Univ.
Nukuikita-machi, koganei-shi,
Tokyo, 184, Japan